JOURNAL OF COMPUTATIONAL PHYSICS **142,** 428–444 (1998)
ARTICLE NO. CP985933

Efficient and Highly Accurate Computation of a Class of Radially Symmetric Solutions of the Navier–Stokes Equation and the Heat Equation in Two Dimensions

Henrik O. Nordmark

Department of Mathematics, Universidad Autonoma Metropolitana, Azcapotzalco, Mexico City, Mexico
E-mail: nord@hp9000a1.uam.mx

Received February 28, 1997

In this paper we test several different formulas for the computation of the exact vorticity and angular velocity in certain radially symmetric solutions of the two-dimensional Navier–Stokes equation in vorticity-stream function form. The class of initial conditions for the vorticity considered here has often been used by many authors in the study of vortex methods. However, only in the case of zero viscosity has it been possible to efficiently compute the exact vorticity and velocity at later times. The expressions for the vorticity and angular velocity, given in this paper, enable us to compute these quantities both efficiently and highly accurately for nonzero viscosity. This makes it feasible to obtain reliable error measurements in the study of vortex methods for the Navier–Stokes equation.

1. INTRODUCTION

In the numerical study of vortex methods for the Euler equations or the Navier–Stokes equations, it is desirable to test the method on problems for which the exact solution is known, and for which the smoothness of the solution can be chosen arbitrarily. This enables us to measure the numerical errors exactly, for these test problems, as well as obtaining reasonably reliable estimates of the rates of convergence for the vorticity and the velocity. In practice, such exact solutions can only be obtained for radially symmetric vorticity, at least if we require smooth solutions, either with compact support, or decaying rapidly at infinity. (Note that the Kirchhoff elliptical vortex is an exact solution of the Euler equations, see [4], which is not radially symmetric, but with discontinuous vorticity.) The reader might argue that it is unsuitable to use radially symmetric vorticity in the numerical study of fluid flow, since the convective term vanishes in this case. However, in vortex methods the convective term is always included in the calculations, even in the cases where it is

0021-9991/98 $25.00

known to vanish in the exact solution. In fact, the convective term will not vanish numerically, but will approach zero as the gridsize tends to zero. For the Euler equations, radially symmetric vorticity implies that the vorticity is constant in time, and the flow is circular. In this case, the exact velocity can be obtained as long as the vorticity can be integrated exactly over a disk centered at the origin. The most popular choice for the vorticity is of the form $(1 - r^2)^k$ for $r \leq 1$ and zero otherwise. This choice has the advantage of being easy to integrate with respect to r in the plane, and the smoothness can be easily varied by the choice of k. For example, Beale and Majda [1] used a vorticity distribution of this form with $k = 3$, while the choice $k = 7$, first used by Perlman [8], has subsequently been used by many researchers. It is much more difficult to find exact solutions for the Navier–Stokes equations. Two examples where the exact vorticity can be given in closed form are the radially symmetric solutions of Gaussian type $\omega(r, t) = e^{-kr^2/(1+4\nu kt)}/(1 + 4\nu kt)$ and Chorin's periodic solution $\omega(x, y, t) = 2e^{-2\nu t}\cos(x)\cos(y)$ (see [2]). Even though Chorin's solution is not radially symmetric, the convective term still vanishes, so both of these solutions also satisfy the heat equation. Hence, any linear combination of Gaussian type solutions with different values of k and Chorin's periodic solution is also a solution of both the Navier–Stokes equations and the heat equation. From a strictly mathematical point of view, vortex methods are not applicable to Chorin's periodic vorticity distribution, since it does not decay at infinity. Actually, Chorin used it to test a projection method in [2]. Nevertheless, Fishelov [3] tested her vortex method on precisely this test case, by exploiting the fact that if one uses the initial condition $\omega(x, y, 0) = 2\cos(x)\cos(y)$ inside the square $-2\pi \leq x, y \leq 4\pi$ and zero outside, then the exact vorticity is very close to $2e^{-2\nu t}\cos(x)\cos(y)$ inside the *smaller square* $0 \leq x, y \leq 2\pi$, for small values of νt. While these analytic exact solutions are useful in testing vortex methods, and other numerical methods for the Navier–Stokes equation and the heat equation, it is desirable to test the methods for less smooth initial conditions. In particular, we would like to use the same type of initial condition for the vorticity, as the one which is commonly used for Euler's equation, i.e., $(1 - r^2)^k$ inside the unit disk and zero outside. The exact vorticity cannot be expressed in closed form in this case, and was usually considered too expensive to compute at many points, since it was only given in terms of a double integral in the literature. Therefore, Roberts [9] and Fishelov [3] who studied the discontinuous case $k = 0$, instead measured the error in the second moment of the vorticity $L(t)$, given by $L(t) = \int_{R^2} |\mathbf{x}|^2 \omega(\mathbf{x}, t)\, d\mathbf{x} / \int_{R^2} \omega(\mathbf{x}, t)\, d\mathbf{x}$, which for the exact vorticity satisfies $L(t) = L(0) + 4\nu t$.

In this paper we find that for any non-negative integer value of k both the vorticity and the angular velocity of the flow can be expressed in terms of convergent series in several different ways. We also find asymptotic expansions for the vorticity and the angular velocity which are both very fast to evaluate and highly accurate for small values of νt. Finally, we present numerical tests of most of these formulas, which show that with an appropriate selection of formula, according to the values of νt and r, we have an efficient method of computing the exact vorticity and velocity for this popular test problem with high accuracy. Some of these formulas were used to calculate the errors in the deterministic vortex method in [7].

2. THE BASIC EQUATIONS

The two-dimensional Navier–Stokes equation in the vorticity stream function form is given by

$$\omega_t + \mathbf{u} \cdot \nabla\omega = \nu\Delta\omega, \tag{1}$$

$$\Delta\psi = -\omega, \tag{2}$$

$$u = \psi_y, \qquad v = -\psi_x, \tag{3}$$

$$\operatorname{div}(\mathbf{u}) = 0, \tag{4}$$

where t stands for time, $\mathbf{u} = (u, v)$ is the velocity vector, $\mathbf{x} = (x, y)$ is the position vector, ω is the vorticity, ν is the viscosity coefficient, and ψ is the stream function. If the initial vorticity is radially symmetric, then $\mathbf{u} \cdot \nabla\omega = 0$, so that (1) reduces to the heat equation, i.e.,

$$\omega_t = \nu\Delta\omega = \nu\left(\omega_{rr} + \frac{\omega_r}{r}\right), \tag{5}$$

with the initial condition

$$\omega(r, 0) = f(r), \tag{6}$$

where $r = |\mathbf{x}|$. It is well known that the solution of the initial value problem for the 2-D heat equation is given by

$$\omega(\mathbf{x}, t) = \frac{1}{4\pi\nu t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-|\mathbf{x}-\mathbf{y}|^2/(4\nu t)} f(\mathbf{y})\, d\mathbf{y}. \tag{7}$$

By introducing polar coordinates in (7), and using the integral representation of Bessel functions, we find that the solution of (5), (6) is given by

$$\omega(r, t) = \frac{e^{-r^2/(4\nu t)}}{2\nu t}\int_0^{\infty} e^{-\rho^2/(4\nu t)} I_0(r\rho/(2\nu t)) f(\rho)\rho\, d\rho, \tag{8}$$

where I_0 is the modified Bessel function of order 0. Another way of obtaining the vorticity as a single integral is by inverting the Fourier transform of the vorticity in polar coordinates.

$$\omega(r, t) = 2\pi\int_0^{\infty} J_0(rs)s\hat{\omega}(s, t)\, ds = 2\pi\int_0^{\infty} J_0(rs)se^{-\nu ts^2}\hat{\omega}(s, 0)\, ds, \tag{9}$$

where we have defined the Fourier transform of ω as

$$\hat{\omega}(|\gamma|, 0) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\mathbf{x}\cdot\gamma}\omega(|\mathbf{x}|, 0)\, d\mathbf{x}. \tag{10}$$

The velocity is obtained by first solving (2) for the stream function and then using (3). When the vorticity is radially symmetric, so is the stream function. Therefore (2) can be rewritten as

$$(r\psi'(r))' = -r\omega, \tag{11}$$

from which we obtain

$$(u, v) = \mu(r, t)(-y, x), \tag{12}$$

where $\mu(r,t)$ is the angular velocity given by

$$\mu(r,t) = \frac{1}{r^2}\int_0^r s\omega(s,t)\,ds. \tag{13}$$

The *particle trajectories* $(x(t), y(t))$ can now be expressed as

$$x(t) = x(0)\cos(\theta(r,t)) - y(0)\sin(\theta(r,t)), \tag{14}$$

$$y(t) = x(0)\sin(\theta(r,t)) + y(0)\cos(\theta(r,t)), \tag{15}$$

where

$$\theta(r,t) = \int_0^t \mu(r,s)\,ds. \tag{16}$$

As we shall see, it will be possible to obtain exact expressions for $\omega(r,t)$ and $\mu(r,t)$ in terms of several different convergent series and also in terms of asymptotic expansions for small νt. While it seems impossible to integrate (16) analytically, this doesn't matter much in practice since normally $\nu \ll 1$, so that $\mu(r,t)$ varies very slowly with time. Therefore (16) can be integrated numerically to extremely high accuracy using a high order method, even with quite large time steps, so in practice we are able to calculate the trajectories (14) and (15) to any desired accuracy.

3. VORTICITY

In this paper, we will consider initial conditions of the form

$$\begin{aligned} \omega(r,0) &= (1-r^2)^k \qquad \text{for } r \le 1, \\ &= 0 \qquad\qquad\quad\ \ \text{for } r > 1. \end{aligned} \tag{17}$$

This type of initial conditions has been used by several authors in the study of vortex methods, including Perlman [8], Beale and Majda [1], Fishelov [3], Russo and Strain [10], Strain [11], and Nordmark [6, 7]. However, since

$$r^{2n} = (1-(1-r^2))^n = \sum_{i=0}^{n}\binom{n}{i}(-1)^i(1-r^2)^i, \tag{18}$$

we can generalize the result obtained here to let the vorticity be given by an arbitrary polynomial in r^2 for $r \le 1$. Furthermore, thanks to the Weierstrass theorem, we can approximate any continuous, radially symmetric, initial data with compact support by a polynomial in r^2 with a uniform error less than any $\epsilon > 0$. It is easy to show that then the error at any later time will also be less than ϵ. In practice, however, such a polynomial will be of reasonably low degree only if $f(r)$ is a smooth function of r^2. With the initial condition (17), (8) becomes

$$\omega(r,t) = \frac{e^{-r^2/(4\nu t)}}{2\nu t}\int_0^1 e^{-\rho^2/(4\nu t)} I_0(r\rho/(2\nu t))(1-\rho^2)^k \rho\,d\rho. \tag{19}$$

Although it is possible to evaluate (19) both rapidly and accurately by Gaussian quadrature, even for small values of νt, care must be taken for such values. This is because the integrand is sharply peaked around $\rho = r$, and is essentially zero outside the interval $r - 11\sqrt{\nu t} \le \rho \le r + 11\sqrt{\nu t}$, when νt is small. However, if we want to calculate the angular velocity using (13), we have to evaluate a double integral, which is more expensive to do numerically. A better way is to express the vorticity as a series, which can then be integrated term by term to obtain the angular velocity as a series as well. It turns out that this can be done in several different ways. Asymptotic expansions can also be found. By setting $u = r^2\rho^2/(4\nu^2t^2)$, (19) becomes

$$\omega(r,t) = \frac{\nu t e^{-r^2/(4\nu t)}}{r^2} \int_0^{r^2/4\nu^2t^2} e^{-\nu t u/r^2}(1 - 4\nu^2t^2u/r^2)^k\, I_0(\sqrt{u})\, du. \tag{20}$$

The first series for the vorticity is obtained by repeated integrations by parts in (20), or by using Sonine's first integral [12, p. 373]

$$\omega(r,t) = (4\nu t)^k e^{-(r^2+1)/(4\nu t)} \sum_{n=k+1}^{\infty} \frac{(n-1)!}{(n-1-k)!} r^{-n} I_n(r/(2\nu t)). \tag{21}$$

To prove the convergence of (21), and to estimate the number of terms needed for an accurate evaluation, we need a bound for modified Bessel functions. Using the integral representation

$$I_\nu(x) = \frac{(x/2)^\nu}{\Gamma(\nu+1/2)\Gamma(1/2)} \int_0^\pi e^{x\cos\theta} \sin^{2\nu}\theta\, d\theta, \tag{22}$$

see [12, p. 204], we immediately get

$$|I_n(x)| \le \frac{\sqrt{\pi}|x/2|^n e^x}{(n-1)!}. \tag{23}$$

Hence,

$$\left| \frac{(4\nu t)^k e^{-(r^2+1)/(4\nu t)} r^{-n} I_n(r/(2\nu t))(n-1)!}{(n-1-k)!} \right| \le \frac{\sqrt{\pi}(1/(4\nu t))^{n-k-1} e^{-(r-1)^2/(4\nu t)}}{4\nu t(n-1-k)!}. \tag{24}$$

Therefore, (21) is majorized by $\sqrt{\pi} e^{-(r-1)^2/(4\nu t)}$ times the Taylor series for $e^{1/(4\nu t)}/(4\nu t)$, and thus converges for all $t > 0$ and rapidly when νt is not small. The remainder term is bounded by $e^{(1-(r-1)^2)/(4\nu t)}(4\nu t)^{-n}/n!$, which by Stirling's formula is approximately $e^{1/(4\nu t)}\sqrt{2\pi n}(4\nu t n/e)^{-n}$ for large n. Hence, for small values of νt, the number of terms needed is estimated by $e/(4\nu t)$, since the factor $(4\nu t n/e)^{-n}$ decreases rapidly as n increases beyond $e/(4\nu t)$. In practice, fewer terms are needed, although it is often hard to avoid using too many terms. On the other hand, when $r > 1$ and νt is small enough, no term at all in (21) is needed. In fact, since $|I_n(x)| \le e^x \ \forall x$ we have

$$\begin{aligned} \left| \sum_{n=k+1}^{\infty} \frac{(4\nu t)^k e^{-(r^2+1)/(4\nu t)} r^{-n} I_n(r/(2\nu t))(n-1)!}{(n-1-k)!} \right| &\le (4\nu t)^k e^{-(r-1)^2/(4\nu t)} \sum_{n=k+1}^{\infty} \frac{r^{-n}(n-1)!}{(n-1-k)!} \\ &= \frac{(4\nu t)^k e^{-(r-1)^2/(4\nu t)} k!}{(r-1)^{k+1}} \qquad \text{for } r > 1. \end{aligned} \tag{25}$$

Hence, there is a function $\delta(\nu t)$ such that $\omega(r,t)$ is negligible for $r > 1 + \delta(\nu t)$, and $\delta(\nu t)$ tends to zero as νt tends to zero. This is expected, since $\omega(r,0)$ is zero for $r \geq 1$. For example, if we want ten correct decimals, and $k \leq 10$ then $\delta(0.001) < 0.32$ and $\delta(0.0001) < 0.11$. On the other hand, if $r < 1$, and νt is small, comparison with the exponential series shows that about $e/(4\nu t)$, terms may be needed to evaluate (21) accurately. This is feasible, since all terms in (21) are positive, and since Bessel functions satisfy the recursion formula

$$I_{n-1}(z) = I_{n+1}(z) + \frac{2nI_n(z)}{z}, \tag{26}$$

which must be applied *backwards*. However, a more efficient approach for $r < 1$ and νt small is to rewrite (20) as

$$\begin{aligned}\omega(r,t) = \frac{\nu t e^{-r^2/(4\nu t)}}{r^2}\Bigg(&\int_0^\infty e^{-\nu t u/r^2}\left(1 - \frac{4\nu^2 t^2 u}{r^2}\right)^k I_0(\sqrt{u})\,du \\ &- \int_{r^2/(4\nu^2 t^2)}^\infty e^{-\nu t u/r^2}\left(1 - \frac{4\nu^2 t^2 u}{r^2}\right)^k I_0(\sqrt{u})\,du\Bigg).\end{aligned} \tag{27}$$

The first integral in (27) can be evaluated explicitly (see [12, p. 385]), while the second integral in (27) can be integrated by parts repeatedly. We then get

$$\begin{aligned}\omega(r,t) = &\sum_{n=0}^{k} \frac{k!}{(k-n)!}(-4\nu t)^n L_n(-r^2/(4\nu t)) - (-4\nu t)^k e^{-(r^2+1)/(4\nu t)} \\ &\times \sum_{n=0}^{\infty} \frac{(n+k)!}{n!} r^n I_n(r/(2\nu t)),\end{aligned} \tag{28}$$

where L_n is the Laguerre polynomial of degree n. We should point out here that the first sum in (28) is the *exact* solution of the two dimensional heat equation with initial condition $\omega(r,0) = (1-r^2)^k\ \forall r$. Therefore, we expect this to be a good approximation when $r < 1$ and νt is small enough. Using again the bound $|I_n(x)| \leq e^x$, we get

$$\begin{aligned}\left|(4\nu t)^k e^{-(r^2+1)/(4\nu t)} \sum_{n=0}^{\infty} \frac{(n+k)!}{n!} r^n I_n(r/(2\nu t))\right| &\leq (4\nu t)^k e^{-(r-1)^2/(4\nu t)} \left|\sum_{n=0}^{\infty} \frac{(n+k)!}{n!} r^n\right| \\ &= \frac{(4\nu t)^k e^{-(r-1)^2/(4\nu t)} k!}{(1-r)^{k+1}} \qquad \text{for } r < 1.\end{aligned} \tag{29}$$

Since this bound is identical to the one in (25), except that $r-1$ is replaced by $1-r$, we may neglect the second sum in (28) when $r < 1 - \delta(\nu t)$, where $\delta(\nu t)$ is defined as before. On the other hand, when $k > 0$, the first sum in (28) becomes arbitrarily large as either r or t tends to infinity. This effect increases rapidly with increasing values of k. For example, if $k = 7$, the effects of roundoff errors are important when either $\nu t > 0.1$ or r is greater than about 1.7 (using double precision). One may be tempted to always use (28) when $r = 0$, since all but a finite number of terms are zero in this case, but this will lead to large round-off errors when νt is fairly large. The least favorable case occurs when νt is small and $1 - \delta(\nu t) < r < 1 + \delta(\nu t)$. In this case either (21) or (28) requires a large number of terms, but both are accurate if enough terms are included. An alternative is to use asymptotic expansions, which we will derive below, but first we shall find three more series representations with the help of the Fourier transform. With the initial condition (17), we have

$$\hat{\omega}(s,0) = \frac{2^k k!\, J_{k+1}(s)}{2\pi s^{k+1}}. \tag{30}$$

Therefore, (9) becomes

$$\omega(r,t) = 2^k k! \int_0^\infty e^{-\nu t s^2} J_0(rs) J_{k+1}(s) s^{-k}\, ds. \tag{31}$$

Expanding $J_{k+1}(s)$ in its MacLaurin series and using [12, p. 393, (2)], we get from (31)

$$\omega(r,t) = -k! e^{-r^2/(4\nu t)} \sum_{n=0}^{\infty} \frac{(-4\nu t)^{-(n+1)} L_n(r^2/(4\nu t))}{(n+k+1)!}. \tag{32}$$

If we instead expand $J_0(rs)$ in its Taylor series with respect to r^2, centered at $r^2 = 0$ and $r^2 = 1$, respectively, and integrate term by term, we obtain the corresponding Taylor series for the vorticity, see [12, pp. 393, 396],

$$\omega(r,t) = \frac{-e^{-1/(4\nu t)}}{k+1} \sum_{n=0}^{\infty} \frac{{}_1F_1(k-n+1; k+2; 1/(4\nu t)) r^{2n}}{n!(-4\nu t)^{n+1}}, \tag{33}$$

$$= \frac{1}{k+1} \sum_{n=0}^{\infty} \frac{(1-r^2)^n {}_2F_2((k+n+2)/2, (k+n+3)/2; k+2, k+n+2; -1/(\nu t))}{n!(4\nu t)^{n+1}}, \tag{34}$$

where ${}_1F_1$ is the confluent hypergeometric function and ${}_2F_2$ is a generalized hypergeometric function. When $n > k$, ${}_1F_1(k-n+1; k+2; 1/(4\nu t)) = L_{n-k-1}^{(k+1)}(1/(4\nu t))$. Otherwise, it may be expressed as a combination of polynomials in νt and $e^{1/(4\nu t)}$. However, (33) and (34) are not very useful for computational purposes, and will not be considered further. We shall just note that

$$\omega(1,t) = \frac{{}_2F_2((k+2)/2, (k+3)/2; k+2, k+2; -1/(\nu t))}{4\nu t(k+1)}, \tag{35}$$

which for $k = 0$ reduces to

$$\omega(1,t) = \frac{(1 - e^{-1/(2\nu t)} I_0(1/(2\nu t)))}{2}. \tag{36}$$

Since

$$L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k n! x^k}{(k!)^2 (n-k)!}, \tag{37}$$

$|L_n(x)|$ is bounded by $(1+|x|)^n$. Hence, by comparison with an exponential series, (32) converges $\forall x$. However, if x is large compared to n, then $L_n(x)$ is dominated by the term $x^n/n!$. Therefore, if νt is very small, the series (32) contains very large terms with alternating signs, which makes its evaluation impossible. Indeed, if $k = 0$, $r = 0$, and $\nu t = 0.02$, then the eleventh term is about $3 \cdot 10^4$ in absolute value. Then the error in the sum will still be less than 10^{-10}, if we use double precision. However, if we reduce νt to 0.01, then the size of the largest term increases to about $5.7 \cdot 10^9$, causing a significant error even in double precision. Finally, if we take $\nu t = 0.005$, the largest term is around 10^{20}, giving a useless result. If we increase k, the limiting value of νt for which (32) can be used, can be taken slightly smaller. In particular, $\nu t = 0.01$ is acceptable in double precision for $k = 7$ (see Table 3). Note that when νt is large enough, (32) provides the simplest, and as we shall see in the numerical section, one of the fastest ways of calculating the vorticity.

4. ASYMPTOTIC EXPANSIONS FOR THE VORTICITY

Since the evaluation of the vorticity using either (21) or (28) is relatively slow for small νt, in a neighborhood of $r = 1$, we seek an asymptotic expansion. Let

$$\omega(r,t) = \left(\frac{1+\operatorname{erf}((1-r)/(2\sqrt{\nu t}))}{2}\right)\sum_{n=0}^{k}\frac{k!}{(k-n)!}(-4\nu t)^n L_n(-r^2/(4\nu t)) + \tilde{\omega}(r,t), \tag{38}$$

where $\tilde{\omega}(r,t)$ is a correction term which tends to zero as νt tends to zero. This is motivated by the facts that $\sum_{n=0}^{k}\frac{k!}{(k-n)!}(-4\nu t)^n L_n(-r^2/(4\nu t))$ is the exact solution of the two-dimensional heat equation with initial value $(1-r^2)^k$ (r unrestricted), and that $(1+\operatorname{erf}((1-r)/(2\sqrt{\nu t})))/2$ tends to $H(1-r)$, where H is the Heaviside function, as νt tends to zero. Plugging (38) into (5), we find that $\tilde{\omega}(r,t)$ satisfies the inhomogeneous heat equation

$$\tilde{\omega}_t - \tilde{\omega}_{rr} - \frac{\tilde{\omega}_r}{r} = e^{-(r-1)^2/(4\nu t)} \times \sum_{n=0}^{k}\frac{(-4\nu t)^n k!((1+4n)L_n(-r^2/(4\nu t)) - 4nL_{n-1}(-r^2/(4\nu t)))}{(k-n)!}. \tag{39}$$

We look for a *formal* series solution of (39)

$$\tilde{\omega}(r,t) = \frac{e^{-(r-1)^2/(4\nu t)}}{\sqrt{\pi}}\sum_{n=0}^{\infty}(\nu t)^{n+(1/2)} q_n(r), \tag{40}$$

and plug this into (39). This leads to the following sequence of ODEs, with the condition that $q_n(r)$ is bounded at $r = 1$,

$$(1-3r-2nr)q_n(r) + 2r(1-r)q_n'(r) = -2q_{n-1}'(r) - 2rq_{n-1}''(r) - \sum_{i=0}^{k-n}(1+4i)\, d_{n,2i} r^{2i}, \tag{41}$$

where

$$d_{n,2i} = \frac{(-4)^n k!(-1)^i (i+n)!}{i!^2 n!(k-i-n)!}, \tag{42}$$

$$d_{n,2i+1} = 0. \tag{43}$$

In (41), we take $q_{-1}(r)$ to be 0. It turns out that the solutions $q_n(r)$ of (41), which are bounded at $r = 1$ are polynomials when $n < k$ but not when $n \geq k$. In the latter case $q_n(r)$

is a rational function of $\sqrt{r}$ with the form

$$q_{n+k}(r) = \frac{p_n(\sqrt{r})}{r^{n+1/2}(1+\sqrt{r})^{2n+1+k}} \qquad \text{for } n \geq 0, \tag{44}$$

where the p_n's are polynomials satisfying

$$(x-1)p_0'(x) + (1+k)p_0(x) - (1+x)^k(2q_{k-1}'(x) - (-4)^k k!) = 0 \tag{45}$$

$$\begin{aligned} 2x(x^2-1)p_n'(x) &+ 2(2n + (2n+k+1)x - (2n-(k+1))x^2)p_n(x) - ((1-2n)^2 \\ &+ (5 - 3k - 18n + 4nk + 16n^2)x + (k-2+4n)^2x^2)p_{n-1}(x) + x(1+x)(4n-3 \\ &+ (8n+2k-5)x)p_{n-1}'(x) - (x^2(1+x)^2)p_{n-1}''(x) = 0, \qquad n > 0. \end{aligned} \tag{46}$$

Hence, we get

$$\begin{aligned} \omega(r,t) \sim &\left(\frac{1+\operatorname{erf}((1-r)/(2\sqrt{\nu t}))}{2}\right)\sum_{n=0}^{k}\frac{k!}{(k-n)!}(-4\nu t)^n L_n(-r^2/(4\nu t)) \\ &- \frac{e^{-(r-1)^2/(4\nu t)}}{\sqrt{\pi}}\left(\sum_{n=0}^{k-1}(\nu t)^{n+1/2}q_n(r) + (\nu t)^k \sum_{n=0}^{\infty}\frac{(\nu t/r)^{n+1/2}p_n(\sqrt{r})}{(1+\sqrt{r})^{2n+1+k}}\right). \end{aligned} \tag{47}$$

For $k = 0$, the situation is simpler, and we can find $p_n(x)$ explicitly in terms of $p_{n-1}(x)$

$$p_0(x) = 1, \tag{48}$$

$$p_n(x) = \frac{(x+x^2)p_{n-1}'(x) + (1-2n)(1+2x)p_{n-1}(x) - b_n(1+x)^{2n}}{x-1}, \tag{49}$$

where

$$b_0 = 1,\, b_n = \frac{4^{-n}}{n!}\prod_{j=1}^{n}(2j-3)(2j+1) = -\frac{(2n+1)!(2n-3)!}{2^{4n-2}n!^2(n-2)!} \qquad \text{for } n \geq 2. \tag{50}$$

For example,

$$p_1(x) = \frac{1+3x}{4}, \tag{51}$$

$$p_2(x) = \frac{9 + 45x + 75x^2 + 15x^3}{32}, \tag{52}$$

$$p_3(x) = \frac{75 + 525x + 1470x^2 + 1890x^3 + 735x^4 + 105x^5}{128}. \tag{53}$$

This follows from the fact that for $k = 0$ we have

$$\omega_r(r,t) = \frac{-e^{-(r^2+1)/(4\nu t)}I_1(r/(2\nu t))}{2\nu t}. \tag{54}$$

If we now differentiate the asymptotic expansion (47), and set it equal to the known asymptotic expansion for $-e^{-(r^2+1)/(4\nu t)}I_1(r/(2\nu t))/(2\nu t)$ (see [12, p. 203, (3)]), we obtain (49).

In general, for any non-negative integer value of k, it turns out that the $(k+1)$st derivative of ω with respect to r is given by a single term, but there seems to be no such direct recursion formula for $q_n(x)$ and $p_n(x)$ for other values of k. Instead, we set

$$q_n(x) = \sum_{i=0}^{2(k-n)-1} a_{n,i} x^i \qquad \text{and} \qquad p_n(x) = \sum_{i=0}^{\max(k,2n-k-1)} c_{n,i} x^i, \tag{55}$$

and obtain the following recursion formulas for the coefficients

$$a_{n,i} = \frac{a_{n,i-1}(1+2i+2n) - a_{n-1,1+i}2(1+i)^2}{1+2i} - d_{n,i},$$
$$\text{for } n = 0, \ldots, k-1 \text{ and } i = 0, \ldots, 2k-2n \tag{56}$$

$$c_{0,i-1} = \frac{k!(2a_{k-1,1} + (-4)^k k!)}{(i+k)(i-1)!(1-i+k)!} + \frac{i c_{0,i}}{i+k}, \qquad \text{for } i = k+1, \ldots, 1 \tag{57}$$

$$\begin{aligned} 2(2n-i)c_{n,i} &= ((i-k-4n)^2 c_{n-1,i-2} + (1+2i+2i^2-k-2ik-6n-12in \\ &\quad + 4kn + 16n^2)c_{n-1,i-1} + (1+i-2n)^2 c_{n-1,i} + 2(1-i-k+2n)c_{n,i-2} \\ &\quad - 2(1+k+2n)c_{n,i-1}) \qquad \text{for } n > 0 \text{ and } i = 0, \ldots, \max(k, 2n-k-1). \end{aligned} \tag{58}$$

Because of the complexity of the asymptotic expansion, it does not seem possible to find the remainder term or an error bound. The smallest term can be found numerically when k, r, and νt are given, so in practice we have to find a range of values of r and νt such that the smallest term is less than the acceptable error. This criterion should only be a guideline, since there is no guarantee that the error is smaller than the first neglected term. To be safe, we should compare the values obtained using the asymptotic expansion with those obtained by using either (21) or (28). We then find that for $\nu t = 0.05$ the error is usually considerably larger than the first term neglected, but when $\nu t \leq 0.01$ the error is approximately equal to the first term neglected, while the terms are decreasing. We also find that for $k = 7$, and $\nu t \leq 0.01$, one term of the last sum in (47) is needed to get an error less than 10^{-10}, while if $k = 0$ five terms are needed. Since larger k means greater smoothness, five terms should give an error less than 10^{-10} when $\nu t \leq 0.01$ for any k. The expansion is obviously not valid for $r = 0$, so we should avoid using it when r is too close to zero.

5. ANGULAR VELOCITY

The series (21), (28), (33), (34), and (32) can each be integrated according to (13), which gives us the following formulas for the angular velocity,

$$\mu(r,t) = \frac{1 - (4\nu t)^{k+1} e^{-(r^2+1)/(4\nu t)} \sum_{n=k+1}^{\infty} (n!/(n-1-k)!) r^{-n} I_n(r/(2\nu t))}{2(k+1)r^2}, \tag{59}$$

$$\begin{aligned} &= \frac{1}{2} \sum_{n=0}^{k} \frac{k!(-4\nu t)^n L_n^{(1)}(-r^2/(4\nu t))}{(k-n)!(n+1)} + \frac{(-4\nu t)^{k+1} e^{-(r^2+1)/(4\nu t)}}{2(k+1)r^2} \\ &\quad \times \sum_{n=0}^{\infty} \frac{(n+k+1)! r^{n+1} I_{n+1}(r/2\nu t)}{n!}, \end{aligned} \tag{60}$$

$$= \frac{1-e^{-r^2/(4\nu t)}}{2(k+1)r^2} - k! e^{-r^2/(4\nu t)} \sum_{n=1}^{\infty} \frac{(-4\nu t)^{-(n+1)} L_{n-1}^{(1)}(r^2/(4\nu t))}{2n(n+k+1)!}, \tag{61}$$

$$= \frac{-e^{-1/(4\nu t)}}{2(k+1)} \sum_{n=0}^{\infty} \frac{{}_1F_1(k-n+1;\, k+2;\, 1/(4\nu t)) r^{2n}}{(n+1)!(-4\nu t)^{n+1}}, \tag{62}$$

$$= \frac{1}{2(k+1)r^2} \times \sum_{n=0}^{\infty} \frac{(1-(1-r^2)^{n+1})\, {}_2F_2((k+n+2)/2,\, (k+n+3)/2;\, k+2,\, k+n+2;\, -1/(\nu t))}{(n+1)!(4\nu t)^{n+1}}, \tag{63}$$

We also obtain the following asymptotic approximation,

$$\mu(r,t) = \frac{1-\eta(r,t)}{2(k+1)r^2}, \tag{64}$$

where

$$\eta(r,t) \sim \left(\frac{1+\operatorname{erf}((1-r)/(2\sqrt{\nu t}))}{2}\right) \times \left(1 - r^2 \sum_{n=0}^{k} \frac{(k+1)!(-4\nu t)^n L_n^{(1)}(-r^2/(4\nu t))}{(k-n)!(n+1)}\right) - \frac{2(k+1)e^{-(r-1)^2/(4\nu t)}}{\sqrt{\pi}} \times \left(\sum_{n=0}^{k} (\nu t)^{n+1/2} g_n(r) + (\nu t)^{k+1} \sum_{n=0}^{\infty} \frac{(\nu t/r)^{n+(1/2)} h_n(\sqrt{r})}{(1+\sqrt{r})^{2n+2+k}}\right), \tag{65}$$

where $g_n(r)$ and $h_n(r)$ are polynomials which satisfy the recursion formulas

$$g_n(x) = \frac{\sum_{i=0}^{k-n} (d_{n,2i}/2(i+1)) x^{2i+2} - 2x g_{n-1}(x) - 2g'_{n-1}(x)}{1-x}, \tag{66}$$

$$h_0(x) = \frac{2x(xp_0(x) + (1+x)^{1+k} g'_k(x))}{x-1}, \tag{67}$$

$$h_n(x) = \frac{(1-2n+(1-k-4n)x)h_{n-1}(x) + 2x^2 p_n(x) + (x+x^2)h'_{n-1}(x)}{x-1}. \tag{68}$$

If we set

$$g_n(x) = \sum_{i=0}^{2(k-n)+1} \alpha_{n,i} x^i \qquad \text{and} \qquad h_n(x) = \sum_{i=0}^{\max(k+1,2n-k)} \gamma_{n,i} x^i, \tag{69}$$

we obtain the following recursion formulas for the coefficients,

$$\alpha_{n,i} = \alpha_{n,i-1} - 2(1+i)\alpha_{n-1,i+1} + \beta_{n,i} - 2a_{n-1,i-1} \quad \text{for } n = 0, \ldots, k \text{ and } i = 0, \ldots, 2(k+1)-2n \tag{70}$$

$$\gamma_{0,i} = \gamma_{0,i-1} - 2c_{0,i-2} - \frac{2\alpha_{k,1}(k+1)!}{(i-1)!(2-i+k)!} \qquad \text{for } i = 1, \ldots, k+2 \tag{71}$$

$$\gamma_{n,i} = (k-i+4n)\gamma_{n-1,i-1} + (2n-1-i)\gamma_{n-1,i} + \gamma_{n,i-1} - 2c_{n,i-2} \quad \text{for } n > 0 \text{ and } i = 0, \ldots, \max(k+1, 2n-k), \tag{72}$$

where

$$\beta_{n,i} = \frac{d_{n,i-2}}{i} \qquad \text{for } i > 0, \tag{73}$$

$$\beta_{n,0} = 0 \qquad \text{for } n > 0, \tag{74}$$

$$\beta_{0,0} = -\frac{1}{2(k+1)}. \tag{75}$$

6. NUMERICAL RESULTS

We now compare the results of the different ways of calculating the vorticity and angular velocity, as well as the average CPU time required per function evaluation. We consider the formulas (19), (21), (28), (32), and (47) to calculate the vorticity and (59), (60), (61), and (64) to calculate the angular velocity. We take $k = 7$ and three different values of νt. A small value $\nu t = 0.0001$, an intermediate value $\nu t = 0.01$, and a "large" value $\nu t = 1$. When $\nu t = 0.0001$, (32) and (61), i.e., the infinite series of Laguerre polynomials, cannot be used because of the catastrophic round-off errors mentioned earlier. When $\nu t = 0.01$, all the above formulas can be used and finally, when $\nu t = 1$ we cannot use the asymptotic expansions (47) and (64). Since the values of vorticity and angular velocity calculated from (21) and (59) are accurate for all values of νt, we use these as benchmark values. Looking at Table 1, we see that the values of the vorticity obtained using (19), (28), and (47) differ very little from those obtained from (21). The difference is less than 10^{-10}, except for (28) when $r > 1.5$. This is caused by round-off error in (28) for large r. The situation is similar for the angular velocity. The evaluation of the integral formula for the vorticity (19) is made by a 21-point Gaussian quadrature applied to each of two subintervals $0 \le \rho \le r$ and $r \le \rho \le 1$ when $r < 1$, and to the single intervals $0 \le \rho \le 1$ when $1 \le r \le 1.1$. When $r > 1.1$, the vorticity is negligible, so no calculation is made in this case. Table 2 gives the CPU time in milliseconds on a SPARCII, for different values of r and $\nu t = 0.0001$, per evaluation of both vorticity and angular velocity, using corresponding pairs of formulas, except in the case of (19), which has no corresponding formula for the angular velocity. Comparing the CPU times for the different methods, we see that the asymptotic expansions are clearly fastest, requiring only at most 0.3 milliseconds per evaluation of vorticity and angular velocity. The evaluation of (19) using Gaussian

TABLE 1
Vorticity Using (19), (21), (28), (47) and Angular Velocity Using (59), (60), (64) for νt = 0.0001 and k = 7

r	$\omega(r,t)$ (21)	Difference from (21) using: (28)	(47)	(19)	$\mu(r,t)$ (59)	Difference (59): (60)	(64)
.2	.7498069721	$-.10 \cdot 10^{-11}$	$-.10 \cdot 10^{-11}$	$.58 \cdot 10^{-10}$	.4342353714	$.16 \cdot 10^{-11}$	$.16 \cdot 10^{-11}$
.4	.2952293471	$.29 \cdot 10^{-12}$	$.29 \cdot 10^{-12}$	$.29 \cdot 10^{-11}$	.2933073978	$-.97 \cdot 10^{-13}$	$-.97 \cdot 10^{-13}$
.6	.0444363972	$-.18 \cdot 10^{-13}$	$-.18 \cdot 10^{-13}$	$-.16 \cdot 10^{-11}$	.1686280396	$.20 \cdot 10^{-14}$	$.20 \cdot 10^{-14}$
.8	.0008431592	$.99 \cdot 10^{-15}$	$.99 \cdot 10^{-15}$	$-.46 \cdot 10^{-13}$	.0976255826	$.43 \cdot 10^{-15}$	$.43 \cdot 10^{-15}$
1.0	.0000000002	$-.53 \cdot 10^{-14}$	$-.26 \cdot 10^{-15}$	0	.0625000000	$.15 \cdot 10^{-15}$	$.76 \cdot 10^{-16}$
1.2	.0000000000	$-.20 \cdot 10^{-13}$	0	0	.0434027778	$-.22 \cdot 10^{-14}$	0
1.4	.0000000000	$-.86 \cdot 10^{-12}$	0	0	.0318877551	$-.23 \cdot 10^{-13}$	0
1.6	.0000000000	$-.54 \cdot 10^{-9}$	0	0	.0244140625	$-.20 \cdot 10^{-10}$	0
1.8	.0000000000	$.30 \cdot 10^{-8}$	0	0	.0192901235	$.13 \cdot 10^{-9}$	0
2.0	.0000000000	$.45 \cdot 10^{-9}$	0	0	.0156250000	$.21 \cdot 10^{-10}$	0

TABLE 2
Number of Terms Used and CPU Time per Evaluation of Vorticity and Angular Velocity for the Different Methods when νt = 0.0001

	Using (21) and (59)		Using (28), (60)		Using (47), (64)		Using (19)*	
r	# of terms	CPU time (ms)	# of terms	CPU time (ms)	# of terms	CPU time (ms)	# of terms	CPU time (ms)
.2	2832	18.4	8	0.5	16	0.2	42	0.9
.4	2676	17.7	8	0.4	16	0.2	42	1.0
.6	2587	15.8	8	0.4	16	0.2	42	0.9
.8	2549	15.0	8	0.4	16	0.3	42	1.0
1.0	2531	15.4	6913	68.5	16	0.2	21	0.5
1.2	0	0.3	9807	101.1	16	0.3	0	0.0
1.4	0	0.4	13039	131.8	16	0.2	0	0.0
1.6	0	0.4	16603	168.9	16	0.3	0	0.0
1.8	0	0.3	20489	230.9	16	0.2	0	0.0
2.0	0	0.2	24691	223.6	16	0.2	0	0.0

* Only the vorticity is evaluated.

quadrature takes about 1 millisecond when $r < 1$, which is also fast, but only the vorticity is obtained. We see that (21) and (59) are relatively slow when $r \leq 1$ since more than 2500 terms are required. On the other hand, when $r \geq 1.1$ the vorticity is less than 10^{-10}, i.e., effectively zero for the accuracy required. Therefore, zero terms in (21) and one term in (59) are used in this case. Therefore, the CPU time used when $r \geq 1.1$ is at most 0.5 milliseconds, which is due to the overhead of evaluating the error bound (25). If we use (28) and (60), the situation is reversed. Very few terms are needed when $r < 1$, and thousands of terms are needed for $r \geq 1$. Hence, by using (28), (60) when $r < 1$ and (21), (59) when $r > 1$, we get a method which is about as fast as asymptotic expansions except at $r = 1$. However, even at $r = 1$ only about 15 milliseconds are required, using (21) and (59). The results for $\nu t = 0.01$ are given in Tables 3 and 4. In this case, we may also use the pure Laguerre polynomial series (32) and (61), which are found to agree very well with the benchmark values. The other

TABLE 3
Vorticity Using (21), (28), (47), (19), and (32) for νt = 0.01 and k = 7

		Difference from (21) using:			
r	$\omega(r,t)$, (21)	(28)	(47)	(19)	(32)
.0	.7756721701	0	—	$0.46 \cdot 10^{-11}$	$0.58 \cdot 10^{-12}$
.2	.6148025431	$0.51 \cdot 10^{-13}$	$0.51 \cdot 10^{-13}$	$-0.88 \cdot 01^{-13}$	$0.23 \cdot 10^{-12}$
.4	.2975834793	$-0.89 \cdot 10^{-15}$	$-0.89 \cdot 10^{-15}$	$-0.37 \cdot 10^{-12}$	$0.19 \cdot 10^{-13}$
.6	.0805564614	$-0.17 \cdot 10^{-14}$	$-0.17 \cdot 10^{-14}$	$-0.17 \cdot 10^{-12}$	$-0.32 \cdot 10^{-14}$
.8	.0104830715	$-0.98 \cdot 10^{-15}$	$-0.87 \cdot 10^{-15}$	$0.15 \cdot 10^{-12}$	$0.53 \cdot 10^{-15}$
1.0	.0005342296	$-0.75 \cdot 10^{-15}$	$0.17 \cdot 10^{-15}$	$0.28 \cdot 10^{-13}$	$-0.17 \cdot 10^{-15}$
1.2	.0000085006	$0.39 \cdot 10^{-13}$	$0.18 \cdot 10^{-14}$	$-0.12 \cdot 10^{-14}$	$0.30 \cdot 10^{-16}$
1.4	.0000000343	$0.79 \cdot 10^{-12}$	$-0.12 \cdot 10^{-15}$	$-0.92 \cdot 10^{-15}$	$0.16 \cdot 10^{-16}$
1.6	.0000000000	$-0.15 \cdot 10^{-10}$	$-0.27 \cdot 10^{-14}$	$-0.32 \cdot 10^{-15}$	$-0.30 \cdot 10^{-10}$
1.8	.0000000000	$0.83 \cdot 10^{-10}$	$-0.57 \cdot 10^{-14}$	$-0.29 \cdot 10^{-16}$	$-0.49 \cdot 10^{-14}$
2.0	.0000000000	$-0.42 \cdot 10^{-9}$	$0.53 \cdot 10^{-13}$	0	0

TABLE 4
Angular Velocity Using (59), (60), (64), and (61) for $\nu t = 0.01$ and $k = 7$

		Difference from (59) using:		
r	$\mu(r, t)$, (59)	(60)	(64)	(61)
.0	.3878360850	0	—	$0.29 \cdot 10^{-12}$
.2	.3461960550	$-0.10 \cdot 10^{-12}$	$-0.10 \cdot 10^{-12}$	$-0.91 \cdot 10^{-13}$
.4	.2510730684	$0.50 \cdot 10^{-15}$	$0.56 \cdot 10^{-15}$	$0.16 \cdot 10^{-14}$
.6	.1581867905	$0.19 \cdot 10^{-15}$	$0.19 \cdot 10^{-15}$	$0.22 \cdot 10^{-15}$
.8	.0966429675	$-0.89 \cdot 10^{-15}$	$-0.82 \cdot 10^{-15}$	$-0.69 \cdot 10^{-16}$
1.0	.0624706520	$-0.25 \cdot 10^{-15}$	$-0.10 \cdot 10^{-15}$	0
1.2	.0434024890	$0.47 \cdot 10^{-14}$	$0.30 \cdot 10^{-15}$	0
1.4	.0318877543	$0.39 \cdot 10^{-13}$	0	0
1.6	.0244140625	$-0.77 \cdot 10^{-12}$	$-0.14 \cdot 10^{-15}$	$0.47 \cdot 10^{-12}$
1.8	.0192901235	$0.40 \cdot 10^{-11}$	$-0.29 \cdot 10^{-15}$	$0.56 \cdot 10^{-16}$
2.0	.0156250000	$-0.23 \cdot 10^{-10}$	$0.28 \cdot 10^{-14}$	0

methods considered give equally good results, with the exception of (28), when $r \geq 1.8$. In Table 5 we see the CPU times required for the different methods, when $\nu t = 0.01$. The asymptotic expansions are again fastest, followed by the expansions in Laguerre polynomials, (32) and (61). Gaussian quadrature again requires about 1 millisecond to calculate the vorticity only. We also see that the benchmark formulas (21), (59) are evaluated much faster even for $r \leq 1$ since at most 83 terms are now required. We finally consider $\nu t = 1$ (see Table 6). In this case, νt is far too large to use asymptotic expansions. We also notice that the use of the pair (28), (60) results in round-off errors as large as 10^{-4} even though double precision is used. Therefore these formulas are not recommended for such large values of νt. On the other hand, the other methods agree extremely well with one another, the difference being less than 10^{-12}. These good methods are all very fast now, requiring at most 0.7 milliseconds (see Table 7). The expansions in Laguerre polynomials stand out,

TABLE 5
Number of Terms Used and CPU Time per Evaluation of Vorticity and Angular Velocity for the Different Methods when $\nu t = 0.01$

	Using (21), (59)		Using (28), (60)		Using (47), (64)		Using (32), (61)		Using (19)*	
r	# of terms	CPU (ms)	# of terms	CPU (ms)	# of terms	CPU (ms)	# of terms	CPU (ms)	# of terms	CPU (ms)
.2	82	1.6	37	1.5	20	0.3	77	0.6	42	0.7
.4	80	1.5	57	1.7	20	0.3	75	0.6	42	0.8
.6	77	1.4	81	2.0	20	0.3	73	0.5	42	0.9
.8	73	1.4	108	2.2	20	0.3	68	0.5	42	0.9
1.0	67	1.3	138	1.9	20	0.3	63	0.5	21	0.5
1.2	61	1.3	172	2.3	20	0.3	60	0.4	21	0.5
1.4	53	1.1	209	2.9	20	0.3	52	0.4	21	0.6
1.6	46	1.1	249	3.3	20	0.3	6	0.1	21	0.6
1.8	41	1.1	292	3.3	20	0.3	6	0.1	21	0.6
2.0	0	0.1	338	3.4	20	0.3	6	0.1	21	0.6

* Only the vorticity is evaluated.

TABLE 6
Vorticity Using (21), (28), (32), (19) and Angular Velocity Using (59), (60), (61) for νt = 1 and k = 7

	$\omega(r, t)$	Difference from (21) using:			$\mu(r, t)$	Difference from (59):	
r	(21)	(28)	(32)	(19)	(59)	(60)	(61)
.0	.0304031627	$.27 \cdot 10^{-7}$	$.18 \cdot 10^{-12}$	0	.0152015814	$.13 \cdot 10^{-7}$	0
.2	.0301088259	$-.14 \cdot 10^{-5}$	$.18 \cdot 10^{-12}$	$.42 \cdot 10^{-15}$	.0151278779	$-.58 \cdot 10^{-6}$	$-.22 \cdot 10^{-13}$
.4	.0292428013	$-.79 \cdot 10^{-6}$	$.17 \cdot 10^{-12}$	$.44 \cdot 10^{-15}$	.0149096098	$-.35 \cdot 10^{-6}$	$-.57 \cdot 10^{-14}$
.6	.0278544229	$-.13 \cdot 10^{-5}$	$.16 \cdot 10^{-12}$	$.33 \cdot 10^{-15}$	.0145551002	$-.52 \cdot 10^{-6}$	$-.19 \cdot 10^{-14}$
.8	.0260207196	$-.22 \cdot 10^{-5}$	$.14 \cdot 10^{-12}$	$.33 \cdot 10^{-15}$	.0140775616	$-.95 \cdot 10^{-6}$	$-.13 \cdot 10^{-14}$
1.0	.0238393418	$-.28 \cdot 10^{-5}$	$.12 \cdot 10^{-12}$	$-.22 \cdot 10^{-15}$	.0134941806	$-.70 \cdot 10^{-6}$	$.31 \cdot 10^{-15}$
1.2	.0214199704	$-.13 \cdot 10^{-4}$	$\cdot 10 \cdot 10^{-12}$	$.17 \cdot 10^{-16}$	.0128249720	$-.32 \cdot 10^{-5}$	$-.11 \cdot 10^{-15}$
1.4	.0188752591	$-.10 \cdot 10^{-4}$	$.83 \cdot 10^{-13}$	$.87 \cdot 10^{-16}$	.0120915147	$-.28 \cdot 10^{-5}$	$-.14 \cdot 10^{-15}$
1.6	.0163123370	$.49 \cdot 10^{-5}$	$.64 \cdot 10^{-13}$	$-.83 \cdot 10^{-16}$	.0113156883	$.67 \cdot 10^{-6}$	$.36 \cdot 10^{-16}$
1.8	.0138257399	$-.28 \cdot 10^{-4}$	$.48 \cdot 10^{-13}$	$.13 \cdot 10^{-15}$	.0105185151	$-.52 \cdot 10^{-5}$	$-.11 \cdot 10^{-15}$
2.0	.0114923594	$-.17 \cdot 10^{-4}$	$.34 \cdot 10^{-13}$	$-.14 \cdot 10^{-15}$	.0097191945	$-.24 \cdot 10^{-5}$	$.57 \cdot 10^{-16}$

requiring only 0.1 milliseconds. All values less than 10^{-17} in absolute value are reported as 0.

7. SUMMARY AND CONCLUSIONS

Finally we summarize the ranges of values of r and νt for which each pair of expansions is appropriate.

(1) Equation (21) for the vorticity and (59) for the angular velocity. This pair may be used for any $r > 0$ and any νt, but it is relatively slow for very small values of νt and $r \leq 1$ as a large number of terms are required in this case. Moreover, for small values of νt, we need a subroutine which *directly* calculates the logarithms of the modified Bessel functions

TABLE 7
Number of Terms Used and CPU Time per Evaluation of Vorticity and Angular Velocity for the Different Methods when νt = 1

	Using (21) and (59)		Using (32), (61)		Using (19)*	
r	# of terms	CPU time (ms)	# of terms	CPU time (ms)	# of terms	CPU time (ms)
.2	19	0.7	10	0.1	42	0.6
.4	19	0.7	10	0.1	42	0.6
.6	19	0.7	10	0.1	42	0.6
.8	18	0.7	9	0.1	42	0.6
1.0	19	0.7	10	0.1	21	0.4
1.2	18	0.7	10	0.1	21	0.4
1.4	18	0.7	10	0.1	21	0.4
1.6	18	0.7	10	0.1	21	0.4
1.8	18	0.7	9	0.1	21	0.4
2.0	18	0.7	9	0.1	21	0.4

* Only the vorticity is evaluated.

and gamma function. This is needed to avoid overflow. On the other hand, when $r > 1$ and νt is very small, (21) and (59) agree very well with the initial values and are therefore very fast to evaluate. As we increase νt, (21) and (59) become much faster for $r \leq 1$, as seen in Tables 5 and 7.

(2) Equation (28) for the vorticity and (60) for the angular velocity. To be safe from large round-off errors, this pair should only be used when $r \leq 1$ and $\nu t \leq 0.1$. It is especially useful when $r < 1$ and νt is so small that the second sum in (28) may be neglected and when $r = 0$, in which case (28) and (60) become finite sums. Note, however, that even when $r = 0$, (28) and (60) should not be used when $\nu t > 0.1$.

(3) Relation (47) for the vorticity and (64) for the angular velocity, i.e., the asymptotic expansions. These are both very accurate and very fast for $r > 0$ and νt less than about 0.01.

(4) Equation (32) for the vorticity and (61) for the angular velocity, i.e., the expansions in Laguerre polynomials. These are both very accurate and very fast when νt is greater than about 0.01 for $k = 7$. For smaller values of k, the limiting value of νt is slightly larger, i.e., about 0.016 for $k = 0$. All values of $r \geq 0$ work when νt is large enough. When $r = 0$ and $\nu t > 0.1$, (32) and (61) should be used instead of (28) and (60).

(5) The integral formula (19) for the vorticity. Formula (19) may be evaluated both very fast and very accurately using Gaussian quadrature $\forall r \geq 0$ and $\nu t > 0$, as long as we take into account the fact that the integrand is effectively zero outside the interval $[r - 11\sqrt{\nu t}, r + 11\sqrt{\nu t}]$ for small values of νt. The drawback here is that there is no analogous formula for the angular velocity.

A general recommendation for building a both efficient and simple algorithm would be to use the asymptotic expansions for $\nu t \leq 0.016$, and the expansions in Laguerre polynomials for $\nu t \geq 0.016$. In the special case $r = 0$, (28) and (60) should be used for $\nu t < 0.1$. If we demand ten correct decimals, this is all we need, except for $k = 0$. The asymptotic expansions give an error close to $1.0 \cdot 10^{-10}$ for some values of r when $k = 0$ and $\nu t = 0.016$, when the optimum number of terms are used (i.e., 5–6 terms in this case). This is certainly good enough for most applications, but if even more accuracy is needed for νt in a neighborhood of 0.016, we recommend using the Bessel function series (21) and (59).

ACKNOWLEDGMENTS

The author thanks John Strain and Ole Hald for interesting discussions.

REFERENCES

1. J. T. Beale and A. Majda, Higher order accurate vortex methods with explicit velocity kernels, *J. Comput. Phys.* **58**, 188 (1985).
2. A. J. Chorin, *Math. Comput.* **22** (104), 745 (1968).
3. D. Fishelov, A new vortex scheme for viscous flows, *J. Comput. Phys.* **86**, 211 (1990).
4. H. Lamb, *Hydrodynamics* (Dover, New York, 1945), 6th ed.
5. F. Milinazzo and P. G. Saffman, The calculation of large Reynolds number two-dimensional flow using discrete vortices with random walk, *J. Comput. Phys.* **23**, 380 (1977).
6. H. O. Nordmark, Rezoning for higher order vortex methods, *J. Comput. Phys.* **97**, 366 (1991).
7. H. O. Nordmark, Deterministic high order vortex method for the 2-D Navier–Stokes equation with rezoning, *J. Comput. Phys.* **129**, 41 (1996).
8. M. Perlman, On the accuracy of vortex methods, *J. Comput. Phys.* **59**, 200 (1985).

9. S. Roberts, Accuracy of the random vortex method for a problem with non-smooth initial conditions, *J. Comput. Phys.* **58**, 29 (1985).
10. G. Russo and J. Strain, Fast triangulated vortex methods for the 2-D Euler equations, *J. Comput. Phys.* **111**, 291 (1994).
11. J. Strain, Fast adaptive 2D vortex methods, *J. Comput. Phys.* **132**, 108 (1997).
12. G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge Univ. Press, Cambridge, UK, 1966), 1st paperback ed.